

Calabi flow in Riemann surfaces revisited: A new point of view

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1 Introduction

P. Chrusciel's now famous paper [12] shows that the Calabi flow exists and converges to a constant scalar curvature metric in a Riemann surface, assuming the existence of a constant scalar curvature metric in the background. Since there always exists a constant scalar curvature metric thanks to the uniformization theorem in Riemannian surface, Chrusciel's proof appears to be satisfactory for most purposes. One memorable feature of Chrusciel's paper is the strong and somewhat mysterious influence of physics. Given the importance of the Calabi flow in Kähler geometry, a more direct mathematical proof of Chrusciel's theorem is desirable. Inspired by Chrusciel's work, the author has worked on a related problem on the existence of extremal Kähler metrics in any Riemannian surface with boundary since 1994 (cf. [7], [10], [8] for further references). The local theory we developed in the aforementioned papers together with our observation that the Calabi flow decreases several interesting functionals (cf. Section 3), are sufficient to provide a new proof of Chrusciel's theorem. In this paper, we also need to assume the uniformization theorem. At the end of this paper, we will remark on how to remove this assumption.

The purpose of this short note is two-fold: first, in Chrusciel's original paper, the Bondi mass estimate plays a crucial role. The idea of Bondi mass might be clear to people with some physics background, but it is hard to connect to those of us who are less well versed in physics. It is also difficult to find a higher dimensional analogue of "Bondi mass." Thus, a proof without the Bondi mass estimate would be appreciated by mathematicians. The second, and the most important point is to cast this Calabi flow problem into new light. In numerous instances in the history of mathematics, a new proof to an old problem from a completely different angle has lead to progress in other, often unrelated problems. In general, for a parabolic equation, one first proves the long time existence of the flow; then one argues that the flow converges by sequence of

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times $t_i \rightarrow \infty$. The uniqueness of the limit for different sequences as time approaches ∞ is usually one of the hardest problems. Moreover, there are very few tools available (cf. [17]) to tackle this problem. The recent development on a Riemannian metric in the space of Kähler metrics (cf. [15], [16] and [13] for more references) provides more tools for us to argue on this point¹. Let \mathcal{H} denote the space of Kähler potentials in a fixed Kähler class (any Kähler metric in a fixed Kähler class determines a unique Kähler potential up to some additive constant. Thus, we sometimes use the term "space of Kähler metrics" and "space of Kähler potentials" interchangeably.) Following a program outlined by Donaldson [13], we proved in [8] that the \mathcal{H} is geodesically convex by $C^{1,1}$ geodesic and more importantly it is a metric space (cf. Theorem B and Definition 2 in Section 2).

Definition 1.1. (*Cauchy curve*): Any curve $c(t), 0 \leq t < \infty$ is called a Cauchy curve in \mathcal{H} if for any $t, s \rightarrow \infty$ the distance of $c(t)$ and $c(s)$ in \mathcal{H} approaches 0 uniformly.

For any initial metric, we prove that the curve in \mathcal{H} obtained by the Calabi flow is precisely a Cauchy curve in \mathcal{H} . The uniqueness of the limit by sequences is an easy consequence of this assertion.

Organization: In Section 3, we describe four energy functionals, each of which is non-increasing under the Calabi flow. Lower bounds for these functionals give us some useful integral estimates. In Section 4, we prove the long time existence of the Calabi flow on a Riemann surface via these integral estimates. In Section 5, we prove that for any sequence of $t_i \rightarrow \infty$, there exists a subsequence where the Calabi flow converges to a constant scalar curvature metric. In Section 6, we use a new technique introduced in Section 2 to prove the uniqueness of the sequential limit of the Calabi flow as $t \rightarrow \infty$. Moreover, we also give an analytic proof of convergence of the Calabi flow to a constant scalar curvature metric (We don't need to adjust the flow by some conformal transformation). In Section 7, we discuss some interesting questions for future study.

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¹For convenience of the readers, we briefly discuss these new developments in section 2.

2 Summary of recent developments in the space of Kähler potentials

Let (V, ω_0) be an n -dimensional Kähler manifold. Mabuchi ([15]) in 1987 defined a Riemannian metric on the space of Kähler metrics, under which it becomes (formally) a non-positive curved infinite dimensional symmetric space. Apparently unaware of Mabuchi's work, Semmes [16] and Donaldson [13] re-discovered this same metric again from different angles. Set

$$\mathcal{H} = \{\varphi | \omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ on } V\}.$$

Clearly, the tangent space of \mathcal{H} is $C^\infty(V)$ if we assume that everything is smooth. For any vector ψ in the tangent space $T_\varphi\mathcal{H}$, we define the length of this vector as

$$\|\psi\|_\varphi^2 = \int_V \psi^2 \omega_\varphi^n d\mu_\varphi,$$

where $d\mu_\varphi$ is the volume element of the corresponding Kähler metric determined by φ . The geodesic equation is

$$\varphi(t)'' - \frac{1}{2}|\nabla\varphi'(t)|_{\varphi(t)}^2 = 0, \quad (2.1)$$

where the derivative and norm in the second term of the left hand side are taken with respect to the metric $\omega_{\varphi(t)}$.

This geodesic equation shows us how to define a connection on the tangent bundle of \mathcal{H} . If $\phi(t)$ is any path in \mathcal{H} and $\psi(t)$ is a field of tangent vectors along the path (that is, a function on $V \times [0, 1]$), we define the covariant derivative along the path to be

$$D_t\psi = \frac{\partial\psi}{\partial t} - \frac{1}{2}(\nabla\psi, \nabla\phi')_\phi.$$

The main theorem formally proved in [15] (and later re-proved in [16] and [13]) is:

Theorem A *The Riemannian manifold \mathcal{H} is an infinite dimensional symmetric space; it admits a Levi-Civita connection whose curvature is covariant constant. At a point $\phi \in \mathcal{H}$ the curvature is given by*

$$R_\phi(\delta_1\phi, \delta_2\phi)\delta_3\phi = -\frac{1}{4}\{\{\delta_1\phi, \delta_2\phi\}_\phi, \delta_3\phi\}_\phi,$$

where $\{ , \}_\phi$ is the Poisson bracket on $C^\infty(V)$ of the symplectic form ω_ϕ ; and $\delta_1\phi, \delta_2\phi \in T_\phi\mathcal{H}$. Then the sectional curvature is non-positive, given by

$$K_\phi(\delta_1\phi, \delta_2\phi) = -\frac{1}{4}\|\{\delta_1\phi, \delta_2\phi\}_\phi\|_\phi^2.$$

We will skip the proof later, interested readers are referred to [15] or [16] and [13] for the proof.

The subject has been quiet since the early pioneer work of Mabuchi (1987) and Semmes (1991). The real breakthrough came in the beautiful paper by Donaldson [13] in 1996, where he outlines the connection between this Riemannian metric in the infinite dimensional space \mathcal{H} and the traditional Kähler geometry, via a series of important conjectures and theorems. In 1997, following his program, the author proves some of his conjectures:

Theorem B [9] *The following statements are true:*

1. *The space of Kähler potentials \mathcal{H} is convex by $C^{1,1}$ geodesic. More precisely, if $\varphi_0, \varphi_1 \in \mathcal{H}$, then there exists a unique geodesic path $\varphi(t)$ ($0 \leq t \leq 1$) connecting these two points, such that the mixed covariant derivatives of $\varphi(t)$ are uniformly bounded from above.*
2. *\mathcal{H} is a metric space. In other words, the infimum of the lengths of all possible curves between any two different points in \mathcal{H} is strictly positive.*

Definition 2.1. *For any two Kähler metrics g_1 and g_2 in \mathcal{H} , define the distance $d(g_1, g_2)$ to be the length of geodesic connecting them.*

Remark 2.2. *One can complete \mathcal{H} by adding all of the limits of the Cauchy sequence under this distance function.*

In [4], E. Calabi and the author proved the following:

Theorem C [4] *The following statements are true:*

1. *\mathcal{H} is a non-positive curved space in the sense of Alexandrov.*
2. *The length of any smooth curve in \mathcal{H} is decreasing under the Calabi flow unless it is represented by a holomorphic transformation. The distance in \mathcal{H} is also decreasing if the Calabi flow exists for all the time (from $t = 0$ to ∞) for any initial smooth metric.*

3 Various energy functionals in Riemann surfaces

3.1 Notations

In this subsection, we set up some notations for later use. Suppose that $g_0 = \sqrt{-1}F_0 dz d\bar{z}$ is a fixed metric and $g = \sqrt{-1}F dz d\bar{z}$ is any metric in the same Kähler class with g_0 . Then g is necessarily conformal to g_0 with same area.

Suppose now that φ is Kähler distortion potential of g w.r.t. g_0 , while e^{2u} is the conformal factor of g w.r.t. g_0 . Then

$$\sqrt{-1}Fdz \wedge d\bar{z} = \sqrt{-1}F_0dz \wedge d\bar{z} + \sqrt{-1}\partial\bar{\partial}\varphi,$$

and

$$g = e^{2u}g_0 = (1 + \Delta_0\varphi)g_0.$$

Here we use $\Delta, \Delta_0, \Delta_g$ to denote the complex Laplacian operator w.r.t. the local metric $dz \wedge d\bar{z}$, g_0 and g respectively. Then

$$\Delta = \frac{\partial^2}{\partial z \partial \bar{z}}, \quad \Delta_g = \frac{\Delta_0}{e^{2u}}.$$

Let K, K_0 denote the scalar curvature of g, g_0 respectively, and let \underline{K} be the average of the scalar curvature². Then

$$K = -\frac{\Delta(\ln F)}{F} \tag{3.1}$$

$$= \frac{-\Delta_0(\ln(1 + \Delta_0\varphi)) + K_0}{(1 + \Delta_0\varphi)} = \frac{-2\Delta u + K_0}{e^{2u}}. \tag{3.2}$$

The Calabi flow is then defined as

$$\frac{\partial \varphi}{\partial t} = K - \underline{K},$$

or equivalently

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\Delta_0 K}{e^{2u}} = \frac{1}{2} \Delta_g K.$$

Let $H^{p,q}(M, g)$ denote the Sobolev space of functions whose p -th derivatives are L^q integral with respect to the metric g .

Finally, if $M = S^2$, let G denote the group of conformal transformations. Then $\dim G = 3$. Use $\eta(S^2)$ denote its Lie algebra, or the space of all holomorphic vector fields in S^2 .

3.2 Various energy functionals

There are several energy functionals which are decreasing under the Calabi flow. This is quite unusual since most heat flows will just decrease one energy functional. In this subsection, we will introduce four of those functionals (this is not a complete list).

²We shall normalize so that \underline{K} is 1, 0, -1 for S^2 , torus and high genus surface.

1. **Area functional:** Area functional is invariant under the Calabi flow. Denote area functional $A(g)$ as $A = \int_M dg$. Then

$$\frac{dA(g)}{dt} = 2 \int_M \frac{\partial u}{\partial t} dg = \int_M \Delta_g K dg = 0.$$

2. **The Calabi energy** was first introduced by E. Calabi in his famous paper [3]. This functional measures essentially the L^2 distance of a given metric from being a constant scalar curvature metric. Namely,

$$Ca(g) = \int_M (K - \underline{K})^2 dg$$

or

$$\frac{d}{dt} Ca(g(t)) = - \int_M \bar{L}^* L(K) \frac{\partial \varphi}{\partial t} dg, \quad (3.3)$$

where L is the Lichernowicz operator w.r.t. the metric g , *i.e.*, for any function f on M ,

$$L(f) = f_{,zz} dz \otimes dz.$$

Here $f_{,zz}$ is the second covariant derivatives of f of pure type:

$$f_{,zz} = \frac{\partial^2 f}{\partial z^2} - \frac{\partial f}{\partial z} \frac{\partial \log F}{\partial z}. \quad (3.4)$$

And $\bar{L}^* L$ is a fourth order self adjoint operator with nonnegative eigenvalue. An important problem in Kähler geometry is to estimate the lower bound of the first non-zero eigenvalue of this 4th order operator over a family of Kähler metrics (such as the family produced by Calabi flow). Under the Calabi flow, we have

$$\begin{aligned} \frac{d}{dt} Ca(g) &= - \int_M \bar{L}^* L(K) (K - \underline{K}) dg \\ &= - \int_M |L(K)|_g^2 dg \leq 0. \end{aligned}$$

The last equality holds unless $L(K) \equiv 0$, or g is an extremal Kähler metric in Riemann surface. By a theorem of Calabi, K must be a constant everywhere in this case. Therefore, during the flow, the Calabi energy must be strictly decreasing.

3. **The Mabuchi energy** was first introduced by Mabuchi in 1987 to prove the uniqueness of Kähler-Einstein metric when the first Chern class is positive [2]. It is also “famous” since it is only defined through its derivatives and it is hard to manipulate directly. For any smooth curve $\varphi(t) \in \mathcal{H}$, the Mabuchi energy is defined by

$$\frac{d}{dt} Ma(g) = - \int_M (K - \underline{K}) \frac{\partial \varphi}{\partial t} dg. \quad (3.5)$$

Note that the Calabi flow is the L^2 gradient flow of the Mabuchi energy. Consequently, it is also decreasing under the Calabi flow:

$$\frac{d}{dt}Ma(g) = - \int_M (K - \underline{K})^2 dg. \quad (3.6)$$

In a Riemann surface, since there always exists a constant scalar curvature metric, the Mabuchi energy has a uniform lower bound (c.f., [2]). Then (denote the metric along Calabi flow as $g(t)$):

$$Ma(g(t))|_0^T = - \int_0^T \int_M (K - \underline{K})^2 dg dt > -C.$$

The constant $C > 0$ is independent of the time T . In other words, if the flow exists for long time, we have:

$$\int_0^\infty Ca(g(t))dt = \int_0^\infty \int_M (K - \underline{K})^2 dg dt < C. \quad (3.7)$$

Since we also know that the Calabi energy itself is decreasing, this implies that Calabi energy decreases to 0 as t approaches ∞ .

4. **The Liouville energy**³. From the definition of the Calabi energy and the Mabuchi energy, especially the definitions via their first derivatives, I believe that there should exist a third functional F whose definition involves some second order operator \mathcal{P} such that

$$\frac{d}{dt}F(g) = - \int_M \mathcal{P}(K) \frac{\partial \varphi}{\partial t} dg,$$

where

$$\mathcal{P} = \Delta_g + \text{lower order terms}$$

and Δ_g is the complex Laplacian operator w.r.t. metric g . In dimension 2, it turns out that $\mathcal{P} = \Delta_g$ is precisely what it needs, and the functional F above is just the well-known Liouville functional. It will be interesting if one could generalize this to higher dimensional Kähler manifolds.

$$F = \int_M \ln \frac{g}{g_0} (K dg + K_0 dg_0) = \int_M (|\nabla u|_{g_0}^2 + 2K_0 u) dg_0.$$

The derivative of this functional is:

$$\begin{aligned} \frac{1}{2} \frac{dF}{dt} &= \int_M (-2\Delta_0 u + K_0) \frac{\partial u}{\partial t} dg_0 \\ &= \frac{1}{2} \int_M (-2\Delta_0 u + K_0) \Delta_g \frac{\partial \varphi}{\partial t} dg_0 \\ &= \frac{1}{2} \int_M \Delta_g K \frac{\partial \varphi}{\partial t} dg. \end{aligned}$$

³According to [1], the Liouville energy represents the log determinant of the Laplacian operator of conformal metrics g . The research in this direction is very active, see [1], [5], [6] for further references in this topic.

Under the Calabi flow, we have

$$\begin{aligned}\frac{dF}{dt} &= \int_M K \Delta_g K dg \\ &= -\frac{1}{2} \int_M |\nabla K|_g^2 dg \\ &= -\frac{1}{2} \int_M |\nabla K|_{g_0}^2 dg_0 \leq 0.\end{aligned}$$

The last equality holds since g and g_0 are pointwise conformal to each other. A well known fact is that the Liouville functional always has a lower bound (cf. [1]). Thus, if the Calabi flow exists for time T (T might be equal to ∞), then there exists a constant C independent of time T such that

$$\int_0^T \int_M |\nabla K|_g^2 dg dt = \int_0^T \int_M |\nabla K|_{g_0}^2 dg_0 dt < C. \quad (3.8)$$

4 Long time existence of the Calabi flow

In this section, we mainly use the weak compactness theorems we derived in [10] [8] and [11] to show that the Calabi flow exists for all the time. For the convenience to the reader, we re-state the two theorems here:

Theorem 4.1. [10] [8] [11] *Let $\{g_n, n \in \mathbf{N}\}$ be a sequence of conformal metrics in Ω with finite energy and area. Then there exists a subsequence of g_n , a limit metric \underline{g} and a finite set of points $\{p_1, p_2, \dots, p_m\}$ such that $g_n \rightharpoonup \underline{g}$ in $H_{loc}^{2,2}(\Omega \setminus \{p_1, p_2, \dots, p_m\})^4$. Moreover, $E(p_i) \cdot A(p_i) \geq 4\pi^2$ where $E(p_i)$ and $A(p_i)$ represent the amount of energy and area concentrated at point p_i respectively. If $\underline{g} \neq 0$, then \underline{g} has a weak cusp singularity at each point p_i ⁵; the total energy concentration could be improved as $E(p_i) \cdot A(p_i) \geq 16\pi^2$; and the last inequality is sharp.*

Theorem 4.2. (Continued from Theorem 1)[10] [8] [11]. *At each bubble point p_i , there exists a local re-normalization of the metrics $h_n = \pi_n^* g_n$ such that for this new sequence of metrics near p_i , there exists a subsequence $\{h_{n_j}, j \in \mathbf{N}\}$ of $\{h_n\}$, a finite number of bubble points $\{q_1, q_2, \dots, q_l\}$ ($0 \leq l \leq \sqrt{\frac{A(p_i) \cdot K(p_i)}{4\pi^2}}$) with respect to the subsequence of metrics $\{h_{n_j}\}$, a metric \underline{h} in $S^2 \setminus \{\infty, q_1, q_2, \dots, q_l\}$ such that: $h_{n_j} \rightharpoonup \underline{h}$ in $H_{loc}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \dots, q_l\})$. If $\underline{h} \equiv 0$ (vanishing case), then $l \geq 2$ and $z = 0$ is a bubble point of \underline{h} . If \underline{g} has a non-negative weak singular angle at p_i , then a) \underline{g} has a weak cusp singularity at p_i ; b) $\underline{h} \neq 0$ and \underline{h} has only weak cusp singularities (including the singular point at $z = \infty$).*

⁴If $\underline{g} \equiv 0$, the above weak converges means $g_n \rightarrow 0$ locally everywhere except the bubble point.

⁵Around each singular point, taking average of the conformal parameter over each concentric circle, the limit angle (if well defined) of the resulting rotationally symmetric metric is the so called ‘‘weak singular angle’’ of the original metric. If the weak singular angle is 0, it is called a weak cusp singular point. The definition of weak angle could be naturally generalized in the most natural way to the case when the limit metric vanishes.

Suppose that $T \in (0, \infty)$ is the maximum time the flow exists. We want to show that there is a regular metric at time T and the flow can be extended beyond time T . For any point p in M , let $\eta(x)$ be any cut off function on a unit ball centered at p w.r.t. metric g_0 . In other words,

$$\eta(x) = \begin{cases} 1 & \text{if } \text{dist}_{g_0}(x, p) < \frac{1}{2}, \\ 0 & \text{if } \text{dist}_{g_0}(x, p) > 1, \\ \in (0, 1) & \text{otherwise.} \end{cases}$$

For any $\epsilon > 0$, define an ϵ cut off function as $\eta_\epsilon(x) = \eta(\epsilon x)$ which is supported in an ϵ ball $B_\epsilon(p)$. Thus

$$\int_{B_\epsilon(p)} |\nabla \eta_\epsilon|_{g_0}^2 dg_0 = \int_{B_1(p)} |\nabla \eta|_{g_0}^2 dg_0.$$

Define a local area functional as

$$A_\epsilon(p) = \int_M \eta_\epsilon dg = \int_{B_\epsilon(p)} \eta_\epsilon dg.$$

By definition, we have

$$\int_{B_{\frac{\epsilon}{2}}} dg \leq A_\epsilon \leq \int_{B_\epsilon} dg.$$

For any fixed ϵ , we have

$$\begin{aligned} \left| \frac{dA_\epsilon}{dt} \right| &= \left| \int_{B_\epsilon} \eta_\epsilon \triangle_0 K dg_0 \right| = \left| \int_{B_\epsilon} (\nabla \eta_\epsilon \cdot \nabla K)_{g_0} dg_0 \right| \\ &\leq \left(\int_{B_\epsilon} |\nabla \eta_\epsilon|^2 dg_0 \right)^{\frac{1}{2}} \cdot \left(\int_{B_\epsilon} |\nabla K|_{g_0}^2 dg_0 \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{B_\epsilon} |\nabla K|_{g_0}^2 dg_0 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus for any time $t_1 < T$, we have

$$\begin{aligned} \left| \int_{t_1}^T \frac{dA_\epsilon}{dt} dt \right| &\leq C \int_{t_1}^T \left(\int_{B_\epsilon} |\nabla K|_{g_0}^2 dg_0 \right)^{\frac{1}{2}} dt \\ &\leq C \sqrt{T - t_1} \left(\int_{t_1}^T \int_{B_\epsilon(p)} |\nabla K|_{g_0}^2 dg dt \right)^{\frac{1}{2}} \\ &\leq C \sqrt{T - t_1}. \end{aligned}$$

The last inequality holds because of the inequality (3.8). In other words, for any $\epsilon > 0$, we have (for any time $t_1 < t_2 < T$):

$$A_\epsilon(p)|_{t_2} - A_\epsilon(p)|_{t_1} \leq C_1 \sqrt{t_2 - t_1}$$

or

$$\int_{B_{\frac{\epsilon}{2}}} dg(t_2) \leq \int_{B_\epsilon} dg(t_1) + C \sqrt{t_2 - t_1}$$

where C is a constant independent of time t .

Now $\{g(t)|0 < t < T\}$ is a 1-parameter family of conformal metrics with finite Calabi energy and area. Suppose that the compactness fails at least for a subsequence $t_i \rightarrow T$. According to the weak compactness Theorem 4.1, there must exists a finite number of points where the area function has a positive concentration. Suppose p is such a point, and $A(p)$ is the positive area concentration. Then

$$A(p) = \overline{\lim}_{r \rightarrow 0} \overline{\lim}_{t_i \rightarrow T} \int_{B_r(p)} dg(t_i) > 0.$$

On the other hand, choose any $t < T$ and fix it for the time being:

$$\begin{aligned} \overline{\lim}_{t_i \rightarrow T} \int_{B_r(p)} dg(t_i) &\leq \overline{\lim}_{t_i \rightarrow T} \left(\int_{B_{2r}(p)} dg(t) + C\sqrt{t_i - t} \right) \\ &\leq \int_{B_{2r}(p)} dg(t) + C\sqrt{T - t}. \end{aligned}$$

Thus

$$\begin{aligned} A(p) &= \overline{\lim}_{r \rightarrow 0} \overline{\lim}_{t_i \rightarrow T} \int_{B_r(p)} dg(t_i) \int_{B_r(p)} dg(t_i) \\ &< \overline{\lim}_{r \rightarrow 0} \left(\int_{B_{2r}(p)} dg(t) + C\sqrt{T - t} \right) \\ &\leq C\sqrt{T - t}. \end{aligned}$$

Now let $t \rightarrow T$, we have $A(p) \rightarrow 0$, a contradiction! Thus for any sequence $t_i \rightarrow T$, the sequence of metrics $g(t_i)$ converges to a limit metric $g(T)$. In other words, the conformal parameters $u(t)$ of metric $g(t)$ remain uniformly bounded from above and below. It is then not difficult to show that the flow actually converges to a smooth metric $g(T)$ as $t \rightarrow T$. The Calabi flow can be extended further beyond time $t = T$. Therefore, the initial assumption the Calabi flow exists only for a finite time is wrong and the flow actually exists for all the time. We then have

Theorem 4.3. *For any smooth initial metric g_0 , the Calabi flow exists for all the time.*

Remark 4.4. *If one can prove that the Calabi energy is preserved along the Ricci flow (without using the maximum principle), then the same idea of using integral estimates of curvature to get necessary control of the curvature may be applied to the Ricci flow in Riemann surface as well.*

5 Convergence of flow for some sequence $t_i \rightarrow \infty$

Proposition 5.1. *Let $\{g_i\}$ be a sequence of conformal metrics with uniformly bounded area, the Calabi energy and the Liouville energy. Suppose further that one of the following holds:*

$$\lim_{i \rightarrow \infty} \int_M (K_{g_i} - \underline{K})^2 d g_i = 0, \quad (5.1)$$

$$\lim_{i \rightarrow \infty} \int_M |\nabla K_{g_i}|_{g_i}^2 d g_i = 0. \quad (5.2)$$

Then, there exists a subsequence of g_i , a corresponding sequence of conformal transformations π_i and a constant scalar curvature metric g_∞ such that $\pi_i^* g_i$ converge to g_∞ in $H^{2,2}(M, g_0)$ in terms of the conformal factors \tilde{u}_i (If $\chi(M) < 0$, then π_i is trivial.), where $\pi_i^* g_i = e^{2\tilde{u}_i} g_0$.

We will first give a proof based on the weak compactness Theorems 4.1 and 4.2. In the case of torus and surfaces of higher genus, we give a second proof which is more traditional. I prefer the first one because it is more likely to be generalized in higher dimensional manifolds, although the second proof is shorter and cleaner.

Proof. We prove this proposition by using the weak compactness Theorems 4.1 and 4.2. If the compactness fails, then there exists a finite number of bubble points p_1, p_2, \dots, p_m such that g_i has a non-trivial area concentration in each bubble point (we follow notations in Theorem 1 above). Consider two cases: $M = S^2$ or M is a torus or higher genus.

In the first case, $M = S^2$. We can re-normalize the sequence so that $\underline{g} \neq 0$ (by conformal transformations). According to Theorems 1 and 2, each bubble metric has a weak cusp singularity at its singular points. On the other hand, suppose that p is such a bubble point (p might be any of p_1, p_2, \dots, p_m), and D is a small disk centered at p (for convenience, we assume that it is a disk of radius 1). Now suppose the sequence of metrics can be re-written as: $g_i = e^{2u_i} |dz|^2$. Here z is the coordinate variable for disk D . According to our assumption that there is a positive area concentration at $z = 0$ (or point p), thus $\max_{x \in D} u_i(x) \rightarrow \infty$. Without loss of generality, we may assume that $u_i(0) = \max_{x \in D} u_i(x) \rightarrow \infty$. Now re-normalize this sequence of metrics by:

$$\tilde{u}_i(x) = u_i(x \epsilon_i) + \ln \epsilon_i, \quad (5.3)$$

where

$$\epsilon_i = e^{-u_i(0)} \rightarrow 0, \quad \forall |x| < \frac{1}{\epsilon_i}.$$

Note that the metric $\tilde{g}_i = e^{2\tilde{u}_i} |dz|^2$ is just a re-normalization of g_i , thus the scalar curvature and area is not changed! In other words, if we denote the scalar curvature of \tilde{g}_i by \tilde{K}_i , then $\tilde{K}_i(x) = K_{g_i}(\epsilon_i x)$. Thus, for any fixed $R > 1$ and i large enough we have

$$\begin{aligned} \int_{|x| < R} \frac{(\Delta \tilde{u}_i)^2}{e^{2\tilde{u}_i}} |dz|^2 &= \int_{|x| < R} \tilde{K}_i^2 d\tilde{g}_i \\ &= \int_{|x| < \epsilon_i R} K_{g_i}^2 d g_i \\ &\leq \int_D K_{g_i}^2 d g_i < C_1, \end{aligned}$$

and

$$\begin{aligned} \int_{|x|<R} e^{2\tilde{u}_i} |dz|^2 &= \int_{|x|<\epsilon_i R} e^{2u_i} |dz|^2 \\ &\leq \int_D dg_i \leq C_2. \end{aligned}$$

By definition, we have $\tilde{u}_i(x) \leq 0$ for any $|x| < R$. Therefore, it is not difficult to choose a subsequence of \tilde{g}_i which converges in every fixed disk $|x| < R$. Suppose the limit metric is \tilde{g} . Then \tilde{g} is a constant scalar curvature metric (Equation (5.1) or (5.2) implies this) in the Euclidean plane with finite area. Thus, \tilde{g} is a smooth metric in S^2 with positive constant curvature. This contradicts the earlier assertion that any bubble metric must only have weak cusp singularities (since the scalar curvature must be negative near a cusp singular point). Therefore, there is no bubble point after a possible re-normalization of conformal transformation. In other words, there is a subsequence of g_i which converges weakly in $H^{2,2}(S^2)$ (up to conformal transformation group) to a metric g_∞ . Equation (5.1) or (5.2) implies that g_∞ has constant scalar curvature.

In the second case, M is either a torus or surface of higher genus. We can not re-normalize like in S^2 case. There are two cases to handle, the case when $\underline{g} \neq 0$, and the case when $\underline{g} \equiv 0$. In the first case when $\underline{g} \neq 0$, one can argue like in S^2 : the bubble metric if existed, must be a round sphere on one hand; and must have one cusp singularity on the other hand. This is impossible since the scalar curvature near cusp singular point must be negative. Therefore there are no bubble point in the first case and the limiting metric must have constant scalar curvature metric (cf. the assumption (5.1) or (5.2)). In the second case, we have $\underline{g} \equiv 0$. We will show that this is impossible by drawing a contradiction. First, it is a ghost vertex⁶ in the tree decomposition of the limit of $\{g_i\}$. Theorems 4.1 and 4.2 imply that the limit of $\{g_i\}$ can be decomposed in a tree structure where each vertex represents a limit of a subsequence of g_i under a local re-normalization (cf. equation (5.3)). The assumption (5.1) or (5.2) ensures that each vertex (non-ghost) corresponds to a metric with constant scalar curvature and finite area. Therefore, each non-ghost vertex must be a sphere! Consider the total Euler character in the limit tree structure, it should be non-positive since this is torus or high genus surface. On the other hand, the contribution from each non vanishing vertex is always positive (since they are S^2); thus the total Euler character in the remaining ghost vertexes must be strictly negative (actually less than -4π if there is one S^2 in the limit!). Moreover, the total area concentration is 0 at each ghost vertex. Therefore, the total energy concentration in the ghost vertex must be infinite by the Schwartz inequality:

$$(-4\pi)^2 \leq \left(\int_{\Sigma} K_{g_i} \right)^2 dg_i \leq \int_{\Sigma} 1 \cdot \int_{\Sigma} K_{g_i}^2 dg_i,$$

⁶We can apply Theorem 4.2 iteratively to any bubble point arisen from taking weak limit of some subsequence of $\{g_i\}$, we eventually obtain a "bubbles on bubbles" phenomena. The limit of metrics at each stage of "blow-up" is regarded as a "vertex" in the limit tree structure of $\{g_i\}$. If the limit metric vanishes identically, it is then called a "ghost vertex." It was proved in [10] that there are only a finite number of ghost vertexes.

where Σ denotes the total collection of "ghost vertexes." $\int_{\Sigma} K_{g_i}^2 dg_i \rightarrow \infty$ because $\int_{\Sigma} 1 \rightarrow 0$. However, the total energy of the tree structure is finite, this is a contradiction! Thus, g_i must converge to some constant scalar curvature metric in both torus and surfaces of high genus. \square

Now we give the second proof on the case of torus or surface with high genus.

Proof. Write $g_i = e^{2u_i} g_0$. Note that the Calabi energy, Liouville energy and area are uniformly bounded along this sequence of metrics. Then we have

$$\int_M \frac{(-\Delta_0 u_i + K_0)^2}{e^{2u_i}} dg_0 \leq C, \quad (5.4)$$

and

$$-C \leq \int_M |\nabla u_i|_{g_0}^2 dg_0 + \int_M 2K_0 u_i dg_0 \leq C. \quad (5.5)$$

Here C is some uniform constant in this proof, and its value may change from line to line. Since area is fixed along the flow: $\int_M e^{2u_i} dg_0 = \int_M dg_0$. It follows that

$$\int_M u_i dg_0 \leq C.$$

Since the underlying surface is torus or surface of higher genus, we may assume (without loss of generality) that K_0 is a non-positive constant. Combining the previous inequality and the equation (5.5), we obtain

$$\int_M |\nabla u_i|_{g_0}^2 dg_0 \leq C.$$

Following from the Moser-Trudinger Inequality, we have

$$\int_M e^{3u_i} dg_0 \leq C.$$

Combining this with equation (5.4), we arrive at

$$\int_M (-\Delta_0 u_i + K_0)^{\frac{6}{5}} dg_0 \leq C.$$

Note that $\int_M e^{2u_i} dg_0 = \int_M dg_0$. It then follows that

$$\|u_i\|_{H^{2, \frac{6}{5}}(M, g_0)} \leq C.$$

In particular, $\|u_i\|_{L^\infty} \leq C$. In view of Theorems 4.1 and 4.2, there is no concentration point for this sequences. It follows that $\{g_i\}$ always converges by sequence. The condition (5.1) or (5.2) implies that the limit metric has constant scalar curvature metric. \square

Definition 5.2. For any metric g , define $\lambda_1(g)$ to be the first eigenvalue of the complex Laplacian operator with respect to the metric g . This is a well defined map from the space of metrics to the positive real line.

Clearly, $\lambda_1(g)$ is invariant under conformal transformation (where the metric is viewed in a different coordinate, but not re-scaled!).

Proposition 5.3. Let $\{g_i\}$ be a sequence of conformal metrics with fixed area (normalized so that total area is $2\pi|\chi(M)|$ except in a torus where one could normalized area to be any size). Suppose that $g_i = e^{2u_i}g_0$ weakly converge to a constant scalar curvature metric g_∞ in $H^{2,2}(M, g_0)$ in terms of conformal parameter u_i . Then $\lim_{i \rightarrow \infty} \lambda_1(g_i) = \lambda_1 > 0^7$. As an immediate corollary, if $\int_M |\nabla K_{g_i}|_{g_i}^2 dg_i \rightarrow 0$, then $\int_M |K_{g_i} - \underline{K}|^2 dg_i \rightarrow 0$.

Proof. Suppose $g_\infty = e^{2u_\infty}g_0$. Then $u_i \rightarrow u_\infty$ in $H^{2,2}(M, g_0)$. Thus $u_i \rightarrow u_\infty$ in C^α for any $0 < \alpha < 1$. In other words, $e^{2u_i} \rightarrow e^{2u_\infty}$ uniformly. The later in turn implies that first eigenvalues of g_i must converge to that of g_∞ . The last statement of the proposition just follows from the Poincare inequality. \square

Remark 5.4. The first eigenvalue of the evolved metrics converges to a positive constant implies that the Sobolev constant for the evolved metric is uniformly bounded from below.

Theorem 5.5. Let $\{g(t)|0 \leq t < \infty\}$ be a one-parameter family of metrics under the Calabi flow. Then for any sequence of numbers $t_i \rightarrow \infty$, there exists a subsequence (denoted again by $\{t_i\}$) such that $g(t_i)$ converges weakly to a constant scalar curvature metric up to conformal transformation.

Proof. Since the Calabi flow exists for long time, the inequality (3.8) holds for time $T = \infty$:

$$\int_0^\infty \int_M |\nabla K|_g^2 dg dt = \int_0^\infty \int_M |\nabla K|_{g_0}^2 dg_0 dt < C$$

for some constant $C > 0$. Therefore there exists at least a sequence of numbers $t_i \rightarrow \infty$ such that

$$\lim_{t_i \rightarrow \infty} \int_M |\nabla K_{g(t_i)}|_{g(t_i)}^2 dg(t_i) dt = \lim_{t_i \rightarrow \infty} \int_M |\nabla K_{g(t_i)}|_{g_0}^2 dg_0 = 0.$$

Proposition 5.1 implies that there exists a subsequence (denoted again by t_i), a subsequence of conformal transformations π_i and a constant scalar curvature metric g_∞ such that $\pi_i^*g(t_i) \rightarrow g_\infty$ weakly in $H^{2,2}(M, g_0)$. Now Proposition 5.3 implies that

$$\begin{aligned} \lim_{t_i \rightarrow \infty} \lambda_1(g(t_i)) &= \lim_{t_i \rightarrow \infty} \lambda_1(\pi_i^*g(t_i)) \\ &= \lambda_1(g_\infty) > 0, \end{aligned}$$

⁷ $\lambda_1(g_\infty) = 1$ if $M = S^2$.

and

$$\begin{aligned}\lim_{t_i \rightarrow \infty} Ca(g(t_i)) &= \lim_{t_i \rightarrow \infty} Ca(\pi_i^* g(t_i)) \\ &= \lim_{t_i \rightarrow \infty} \int_M |K_{g(t_i)} - \underline{K}|^2 d g(t_i) = 0.\end{aligned}$$

However, the Calabi energy decreases monotonely under the Calabi flow. The above inequality implies that

$$\lim_{t \rightarrow \infty} Ca(g(t)) = \lim_{t \rightarrow \infty} \int_M |K_{g(t)} - \underline{K}|^2 d g(t) = 0.$$

For any sequence $t_i \rightarrow \infty$, Proposition 5.1 again implies that there exists a subsequence (denoted again by t_i), a sequence of conformal transformation π_i , and a constant scalar curvature metric g_∞ such that $\pi_i^* g(t_i) \rightarrow g_\infty$ weakly in $H^{2,2}(M, g_0)$. \square

Notice that the constant scalar curvature metric g_∞ so obtained depends on the sequence chosen. However, all constant scalar curvature metrics in M^2 with same area have the same spectrum, in particular, the same first and second eigenvalues. Therefore, if one denotes $\lambda_1(g_\infty)$, $\lambda_2(g_\infty)$ as the first and second eigenvalues of the limit constant curvature metric, it is then well defined (independent of a sequence chosen). As a matter of fact, on S^2 we have $\lambda_1(g_\infty) = 1$ and $\lambda_2(g_\infty) = 3$. This leads to the following corollary:

Corollary 5.6. *Let $\{g(t) | 0 \leq t < \infty\}$ be a one-parameter family of metrics under the Calabi flow. For any $\epsilon > 0$, for t large enough, the eigenvalues of $g(t)$, either are between $(1 - \epsilon, 1 + \epsilon)$ or are bigger than 2. Let Λ_{first} denote the eigenspace of $g(t)$ corresponds to the eigenvalues between $1 - \epsilon$ and $1 + \epsilon$. Then Λ_{first} converges to the first eigenspace of some constant scalar curvature metric in the limit.*

6 Uniqueness of the limit at $t \rightarrow \infty$ for different sequences

Proposition 6.1. *There exists a small positive number α and a big constant C such that*

$$\int_M (K - \underline{K})^2 d g(t) < C e^{-\alpha t}, \quad \forall t > 0.$$

Proof. Let $\epsilon(t)^2 = \int_M (K - \underline{K})^2 d g(t)$ and let C denote any generic constant. Then $\epsilon(t) \rightarrow 0$ as $t \rightarrow \infty$. We want to show that $\epsilon(t)$ decays exponentially fast (note that $K^{,zz} = \frac{1}{F^2} K_{,zz}$ in the following calculation):

$$\begin{aligned}
& \frac{d}{dt} \int_M (K - \underline{K})^2 dg(t) \\
&= - \int_M K_{,zz} K^{,zz} dg(t) \\
&= - \int_M (\Delta_g K)^2 dg(t) + \int_M \frac{K}{2} |\nabla K|_g^2 dg \\
&= - \int_M (\Delta_g K)^2 dg(t) + \int_M \frac{K - \underline{K}}{2} |\nabla K|_g^2 dg + \frac{K}{2} \int_M |\nabla K|_g^2 dg \\
&\leq - \int_M (\Delta_g K)^2 dg(t) + \frac{1}{2} \left(\int_M (K - \underline{K})^2 dg \right)^{\frac{1}{2}} \cdot \left(\int_M |\nabla K|_g^4 dg \right)^{\frac{1}{2}} + \frac{K}{2} \int_M |\nabla K|_g^2 dg \\
&= - \int_M (\Delta_g K)^2 dg(t) + C\epsilon(t)^{\frac{1}{2}} \times \left(\int_M |\nabla K|_g^4 dg \right)^{\frac{1}{2}} + \frac{K}{2} \int_M |\nabla K|_g^2 dg. \tag{6.1}
\end{aligned}$$

Now we need to consider two cases: $\underline{K} \leq 0$ or $\underline{K} > 0$. The first case is easier, while the second case is more delicate.

Consider the first case. If $\underline{K} \leq 0$, then there exists some positive constant α and C such that

$$\begin{aligned}
& \frac{d}{dt} \int_M (K - \underline{K})^2 dg(t) \\
&\leq - \int_M (\Delta_g K)^2 dg(t) + C\epsilon(t) \times \left(\int_M |\Delta_g K|_g^2 dg \right) - \frac{|\underline{K}|}{2} \int_M |\nabla K|_g^2 dg \\
&\leq -C \int_M (\Delta_g K)^2 dg(t) \leq -\alpha \int_M (K - \underline{K})^2 dg(t)
\end{aligned}$$

for some positive constant $\alpha > 0$. Thus, we have

$$\int_M (K - \underline{K})^2 dg \leq Ce^{-\alpha t}, \quad \text{for any } t > 0.$$

Next we consider the second case. If $\underline{K} > 0$, then our normalization yields that $\underline{K} = 1$. We can rewrite equation (6.1):

$$\begin{aligned}
& \frac{d}{dt} \int_M (K - \underline{K})^2 dg(t) \\
&\leq -(1 - C\epsilon(t)) \int_M (\Delta_g K)^2 dg(t) - \frac{1}{2} \int_M (K - \underline{K}) \Delta_{g(t)} K dg \\
&= -(1 - C\epsilon(t)) \int_M (\Delta_g K)^2 dg(t) + \frac{1}{2} \int_M (\Delta_g K)^2 dg(t) + \frac{1}{2} \int_M (K - \underline{K})^2 dg(t) \\
&= -\left(\frac{1}{2} - C\epsilon(t)\right) \int_M (\Delta_g K)^2 dg(t) + \frac{1}{2} \int_M (K - \underline{K})^2 dg(t). \tag{6.2}
\end{aligned}$$

Since the first eigenvalue of $g(t)$ converges to 1, the above inequality seems to give us little control on the decay rate of $\frac{d}{dt} \int_M (K - \underline{K})^2 dg(t)$. However, by using Kazdan-Warner condition, we can still prove the above estimate in S^2 .

Notice that Kazdan-Warner condition implies

$$\int_M X(K - \underline{K}) d g(t) = \int_M (K - \underline{K}) \triangle_{g(t)} \theta_X d g(t) = 0,$$

where X is any holomorphic vector field in S^2 , and θ_X is the potential of the vector field X with respect to the metric $g(t)$ ⁸. The Kazdan-Warner condition can be re-written as

$$\int_M (K - \underline{K}) \triangle_{g(t)} \theta_X e^{2u} d g_\infty = 0. \quad (6.3)$$

Note that $g(t)$ converges to some constant scalar curvature metric in S^2 in the $H^{2,2}$ sense. This should be enough for our purpose. For t large enough, we can choose a constant scalar curvature metric g_∞ (which may depends on t) such that

$$g(t) = e^{2u} g_\infty$$

and

$$\|u\|_{H^{2,2}(S^2, g_\infty)} \rightarrow 0$$

as $t \rightarrow 0$. This means that the eigenvalue of $g(t)$ converges to those of some constant scalar curvature metric (whose first, second eigenvalues are 1, 3 respectively). As Corollary 1 implies, the eigenspace Λ_{first} of $g(t)$ corresponds to eigenvalues between $1 - \epsilon$ and $1 + \epsilon$ converges to the first eigenspace of g_∞ .

If we decompose $K - \underline{K}$ into two components:

$$K - \underline{K} = \rho + \rho^\perp$$

where

$$\rho \in \Lambda_{\text{first}}(g(t)), \quad \text{and} \quad \rho^\perp \perp \Lambda_{\text{first}}(g(t)).$$

Then the Kazdan-Warner (6.3) condition implies that ⁹:

$$\|\rho\|_{L^2(g(t))} \leq \epsilon \|K - \underline{K}\|_{L^2(g(t))},$$

⁸For any holomorphic vector field X on S^2 , and for any metric g , one can define the potential function θ_X (up to addition of some constants) as the following:

$$\int_M X(\psi) d g = - \int_M \theta_X \triangle_g \psi d g,$$

where ψ is any smooth test function. If g is a constant scalar curvature metric, then θ_X is an eigenfunction of g with eigenvalue 1, i.e.,

$$\triangle_g \theta_X = -\theta_X.$$

Recall $\eta(S^2)$ denote the lie algebra of all holomorphic vector fields on S^2 . Then

$$\{\triangle_g \theta_X \mid \forall X \in \eta(S^2)\} = \{\theta_X \mid \forall X \in \eta(S^2)\}$$

is the first eigenspace of g .

⁹Note that $\{\triangle_{g(t)} \theta_X \mid \forall X \in \eta(S^2)\}$ and $\Lambda_{\text{first}}(g(t))$ both converge to the first eigenspace of same constant scalar curvature metric g_∞ in the limit as $t \rightarrow \infty$. Thus the Kazdan-Warner (6.3) condition implies that the projection of $K - \underline{K}$ into $\Lambda_{\text{first}}(g(t))$ is very small compared to the L^2 norm of $K - \underline{K}$.

where $\epsilon \rightarrow 0$ as $t \rightarrow \infty$. Plugging this into equation (6.2), we have (note that $\underline{K} = 1$ in the following calculation):

$$\begin{aligned}
& \frac{d}{dt} \int_M (K - \underline{K})^2 dg(t) \\
& \leq \left(\frac{1}{2} - C\epsilon(t)\right) \int_M (\Delta_g K)^2 dg(t) + \frac{1}{2} \int_M (K - \underline{K})^2 dg(t) \\
& \leq -\left(\frac{1}{2} - C\epsilon(t)\right) 2^2 (1 - \epsilon) \int_M (K - \underline{K})^2 dg(t) + \frac{1}{2} \int_M (K - \underline{K})^2 dg(t) \\
& \leq -\alpha \int_M (K - \underline{K})^2 dg(t),
\end{aligned}$$

where $\alpha = \left(\frac{1}{2} - C\epsilon(t)\right) 4(1 - \epsilon) - \frac{1}{2}$. Choose ϵ small enough and t large enough (note that $\lim_{t \rightarrow \infty} \epsilon(t) = 0$), we have $\alpha > 0$. It follows that we can easily see the exponential decay of L^2 norm of $K - \underline{K}$:

$$\int_M (K - \underline{K})^2 dg \leq Ce^{-\alpha t}, \quad \text{for any } t > 0.$$

□

The key estimate we used in this calculation is that the spectrum of the evolved metrics $g(t)$ converges to that of a constant scalar curvature metric. All of the constant scalar curvature metrics in S^2 (with same area) are isometric to each other. Thus any geometric norm, such as L^2 norm of ∇K or any higher order derivatives, can be computed and proved to converge exponentially in a similar fashion, just as Chrusciel did in his original paper. Here we want to prove the uniqueness directly using this idea in infinite dimensional space. The point of view one should adopt is that the one parameter family of metrics $\{g(t) | 0 \leq t < \infty\}$ represents a curve in \mathcal{H} . If this curve runs to ∞ in \mathcal{H} in terms of geodesic distance, then Calabi flow diverges. If this curve stays within a bounded domain in \mathcal{H} , then this corresponds to the case that each subsequence converges, but converges to different limit as $t \rightarrow \infty$. However, the case we are in is the best possible: the Calabi flow represents a "Cauchy" sequence under this Riemannian metric. Thus, the limit metric must be unique. To verify this, for any $t_i > s_i \rightarrow \infty$, we want to show the length of curves $\{g(t) | s_i < t < t_i\}$ tends to 0 fast. Denote the distance of two metrics $g(t_i)$ and $g(s_i)$ by $d(g(t_i), g(s_i))$. Then

$$\begin{aligned}
d(g(t_i), g(s_i)) & \leq \int_{s_i}^{t_i} \sqrt{\int_M \left(\frac{\partial g}{\partial t}\right)^2 dg(t)} dt \\
& = \int_{s_i}^{t_i} \sqrt{\int_M (K - \underline{K})^2 dg(t)} dt \\
& \leq \int_{s_i}^{t_i} Ce^{-\alpha t} dt \\
& \leq C \frac{e^{-\alpha s_i} - e^{-\alpha t_i}}{\alpha} \rightarrow 0.
\end{aligned}$$

Thus the limit as $t \rightarrow \infty$ must be unique. We then prove the following

Theorem 6.2. *There exists a unique constant scalar curvature metric $g_\infty \in \mathcal{H}$ such that the Calabi flow converges to this metric exponentially fast in terms of geodesic distance.*

Remark 6.3. *g_∞ could still be a constant scalar curvature metric concentrated on one point. In the following subsection, we will show that this concentration does not occur along the Calabi flow.*

6.1 Convergence of conformal factors e^{2u} as $t \rightarrow \infty$

Theorem 6.4. *For any metric g_0 in Riemann surface, the Calabi flow exists for all the time. For every sequence $t_i \rightarrow \infty$, there exists a subsequence t_i such that $g(t_i)$ converge to a constant scalar curvature metric. And this limiting constant scalar curvature metric is independent of sequence chosen.*

In light of Theorem 6.2 and the remark following it, we only need to show that conformal parameters does not concentrate in any point; or to show it is always bounded.

Proof. We already know that there exists a family of conformal transformations π_i such that $\pi_i^* g(t_i)$ converges to a constant scalar curvature metric g_∞ . Then the set $\{\pi_i | i = 1, 2, \dots\}$ must be compact¹⁰. Otherwise, suppose that $\{\pi_i | i = 1, 2, \dots\}$ is a non-compact family of conformal transformations. By a direct calculation, one should yield:

$$d(g_\infty, \{\pi_i^{-1}\}^* g_\infty) \rightarrow \infty.$$

On the other hand,

$$0 = \lim_{i \rightarrow \infty} d(g_\infty, \pi_i^* g(t_i)) = \lim_{i \rightarrow \infty} d(g(t_i), \{\pi_i^{-1}\}^* g_\infty).$$

By a triangle inequality for distance function, we have

$$\begin{aligned} \lim_{i \rightarrow \infty} d(g_\infty, g(t_i)) &\geq \lim_{i \rightarrow \infty} (-d(\{\pi_i^{-1}\}^* g_\infty, g(t_i)) + d(g_\infty, \{\pi_i^{-1}\}^* g_\infty)) \\ &= \infty. \end{aligned}$$

However, g_∞ is invariant under the Calabi flow, $g(t_i)$ is in the image of g_0 under Calabi flow at time $t = t_i$. By Theorem C, the distance in \mathcal{H} decreases under the Calabi flow. Consequently, we have

$$\infty = \lim_{i \rightarrow \infty} d(g_\infty, g(t_i)) \leq d(g_\infty, g_0) < \infty.$$

This is a contradiction! Thus the set of conformal transformations $\{\pi_i\}$ must be compact and the conformal factor must be bounded. \square

7 Future questions

In this section, we list some problems we think they might be interesting, and at the same time, accessible. Suppose that \mathcal{A} is a subset of the space of subset of \mathcal{H} . Define the diameter of this set as the maximal distance under this metric defined by [15] [16] and [13].

¹⁰This is equivalent to say $|u_i| \leq C$ for some uniform constant C for all $i \geq 1$. Here u_i is the conformal parameter of g_i , i.e., $g_i = e^{2u_i} g_0$.

Question 7.1. *Considering metric in a Riemann surface, suppose $\mathcal{A} = \{g | A(g) < C, Ca(g) < C\}$ and diameter of \mathcal{A} is finite, is \mathcal{A} compact?*

In [10], we give a weak compactness theorem of \mathcal{A} . For a sequence of metrics with finite energy and area, the weak compactness fails because large area concentrates at some isolated points. We ask if this distance of infinite dimensional space necessary approaches to ∞ once the concentration of area occurs in some isolated points.

Question 7.2. *What is the relationship between the distance between any two Kähler metrics in a fixed Kähler class (defined on [15], [16] and [13]) and the Gromov's Hausdorff distance between any two Riemannian metrics. Certainly, the later one is more general, but one wants to understand the difference and similarity when restricted to the space of Kähler metrics. The question above is an attempt in this direction.*

Question 7.3. *Can one prove the lower bound of Mabuchi energy directly (without appealing to the Mabuchi-Bando's theorem)? On a Riemann Surface, the Mabuchi energy takes the form¹¹:*

$$Ma(\varphi) = \int_M \ln(1 + \Delta\varphi)(1 + \Delta\varphi) - \frac{1}{2}\underline{K}|\nabla\varphi|^2 - (K_0 - \underline{K})\varphi$$

for any φ such that $1 + \Delta\varphi > 0$ on M . Here all of the norm are taken w.r.t. the metric g_0 . It is easy to see that the Euler-Lagrange equation for this functional is $K - \underline{K} = 0$, if we recall the formula (3.2) for scalar curvature in this setting. The question we want to ask is if there exists a universal constant $C > 0$ such that for any φ , the following inequality holds:

$$\int_M \ln(1 + \Delta\varphi)(1 + \Delta\varphi) - \frac{1}{2}\underline{K}|\nabla\varphi|^2 - (K_0 - \underline{K})\varphi > -C? \quad (7.1)$$

We know this inequality holds since there always exists a Kähler-Einstein metric in Riemannian surface and a theorem of Mabuchi-Bando implies that the Mabuchi energy bounded from below if there is a Kähler-Einstein metric. The question we ask here is if we can get this inequality without appealing to this theorem of Mabuchi and Bando. If we can prove this in Riemannian surface, then there is at least some hope that we might be able to get a lower bound in higher dimension in some cases without knowledge of Kähler-Einstein metrics.

In a way, this should be similar to Moser-Trudinger-Onofri inequality and one should be able to get this with pure analytic tools.

Question 7.4. *The proof of lower bound of the Liouville energy makes a strong use of Trudinger inequality and it looks quite hard to get a new proof without*

¹¹The explicit formulation of the Mabuchi energy in general Kähler class was first given by Tian, cf. [18].

assuming uniformization theorem. However, it might be a lot easier to prove the same statement when restricted to those conformal metrics with bounded the Calabi energy and area. This will be sufficient for our purpose since the Calabi flow decreases the Calabi energy and preserves the total area.

Conjecture 7.5. *In a topological 2-sphere M , suppose $g_i = e^{2u_i}g_0$ is a family of conformal metrics with bounded area and energy. Suppose the following holds*

1.

$$\lim_{i \rightarrow \infty} \int_M |\nabla K_{g_i}|^2 d g_0 = 0.$$

2. *There exists only one bubble point $p \in S^2$, and $u_i \rightarrow -\infty$ in any compact subset $\Omega \subset M \setminus \{p\}$.*

we conjecture that there exists a flat metric $g_\infty = e^{2u_\infty}g_0$ in $M \setminus \{p\}$ and a subsequence of g_i such that the following holds

$$u_i - c_i \rightharpoonup u_\infty \text{ in } H^{2,2}(\Omega)$$

where Ω is any compact subset of $M \setminus \{p\}$, and $c_i = u_i(q)$ for any fixed point $q \in \Omega \subset M \setminus \{p\}$.

Remark 7.6. *This conjecture implies that M is the standard S^2 . This conjecture, together with the lower bound of the Liouville energy, will re-prove the uniformization theorem via the Calabi flow.*

Remark 7.7. *There is a local version of the above conjecture which had been worked out by Yan Yan Li [14] and it is interesting to compare the two problems.*

Added after the proof: We notice the recent work of M. Struwe, following a similar idea of this paper, giving a more concise proof to the convergence of the Calabi flow from analytic point of view; moreover, he gives a new proof to the Ricci flow in Riemannian surfaces (using the idea of integral estimate only). One key lemma he proves is that the Calabi energy is preserved and eventually improved in the Ricci flow in Riemann surface.

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